DISCRETE MATHEMATICS I

B.MATH 2ND YEAR

END TERM EXAM

Instructions

- Part A contains 10 questions. Each question carries 10 marks each. Answer any 6.
- Part B contains 3 questions. Each question carries 20 marks each. Answer any 2.
- Time limit for the exam is 3 hours.

Notations

- $\mathbb{N} = \{0, 1, 2, \ldots\}$.
- $[n] = \{1, 2, \ldots, n\}$, for $n \in \mathbb{N}$.

Part A (10 marks per question, answer any 6)

1. (a) How many 2×2 magic squares of weight k are there?

A magic square of weight k is a square matrix with entries from $\mathbb{N}$, every row and column of which adds up to k.

(b) Consider an $n \times n$ Latin square L with entries from $[n]$. Prove that L is a magic square. What is the weight of L?

A Latin square is an $n \times n$ matrix with entries from $[n]$, such that no two entries in a row or column are the same.

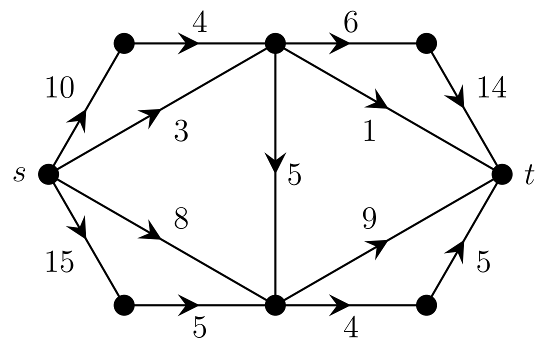
2. The BMath second year students have formed several cliques to study for end semester exams. Every clique consists of 6 students, and every student belongs to exactly 6 cliques.

(a) Prove that, the number of students is equal to the number of cliques.

(b) Is it possible to choose a student from every clique without choosing any student twice?

3. Let G be a finite simple undirected bipartite graph with more than one vertex. Let A_G be the adjacency matrix of G. Prove that, for every $n \geq 1$, the matrix $(A_G)^n$ has a zero entry.

4. Consider the following transportation network with source s and sink t.



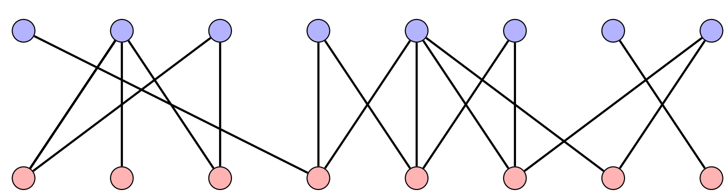
Find a feasible flow in this network with the maximum possible value.

5. Let N be a network. Let (S_1, T_1) and (S_2, T_2) be two $s - t$ cuts in N with minimum capacity. Prove that all edges between $T_1 \setminus T_2$ and $T_2 \setminus T_1$ have zero capacity.

6. Let Q_n be the n-dimensional cube graph. Prove that Q_n has a Hamiltonian cycle for all $n \geq 2$.

Q_k is defined as follows:
- Q_0 is a graph with a single vertex and no edges.
- For all $k \in \mathbb{N}$, the graph Q_{k+1} is the graph constructed by taking two copies of Q_k, then joining every vertex from the first copy to the corresponding vertex in the second copy by an edge.

7. What is the maximum size of a matching in the following bipartite graph?



8. Construct a BIBD with each of the following parameters or prove that they can not be constructed:

(a) $(7, 7, 4, 4, 2)$

(b) $(22, 22, 15, 15, 10)$

9. A Steiner triple system is a BIBD with $k = 3$ and $\lambda = 1$.

(a) How many blocks does a Steiner system with v points have?

(b) Given any point x, how many of those blocks contain x?

10. How many perfect matchings do the following graphs have?

(a) The complete graph with $2n$ vertices, K_{2n}.

(b) The $2n$ length cycle, C_{2n}.

Part B (20 marks per question, answer any 2)

11. (a) Let $\{c_n\}_{n\in\mathbb{N}}$ be a sequence defined by the recurrence relation

$$c_0 = 4, \quad c_1 = 5, \quad c_{n+1} = 3c_{n-1} - 6 \quad \forall\ n \geq 1.$$

Find the value of c_n for all $n \in \mathbb{N}$.

(b) Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence defined by the recurrence relation

$$a_0 = 0, \qquad a_{n+1} = 2(n+1)a_n + (n+1)! \quad \forall\ n \in \mathbb{N}.$$

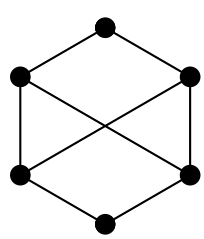
Find the value of a_n for all $n \in \mathbb{N}$.

12. Let $S = [15] \times [30]$. Construct the partial order $\leq'$ on the set S as follows: $(a, b) \leq' (c, d)$ if $a \leq c$ and $b \leq d$.

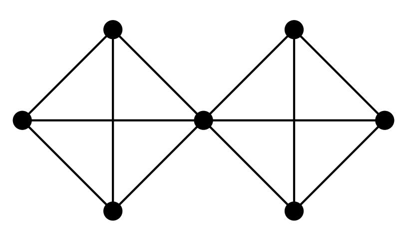
(a) What is the minimum size of a chain decomposition of this poset?

(b) How many antichains are there with the maximum length?

13. How many spanning trees do the following graphs have?



G_1



G_2